UDC 539.3

INTEGRAL EQUATIONS OF CONTACT PROBLEMS FOR THIN-WALLED ELEMENTS

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We present a method of constructing exact solutions of one-dimensional integral equations of special type, to which we reduce the problems of contact between elastic thin-walled elements and a solid, and between the elements themselves. As we know, problems of this type were first formulated by Galin [1] who assumed that the Kirchhoff-Love hypothesis is correct for a thin-walled element. In the present paper we use the model adopted in [2]: the median surface of the thin-walled element (shell, plate or beam), the normal displacements of which obey the above-named hypothesis, is covered with a layer of an elastic Winkler-type support with the pliability coefficient k.

1. Formulation of the typical problem. Let us consider the following contact problem. A hinged beam of length $2a$ (Fig. 1) and flexural rigidity D is acted upon by a force P which impresses into it an absolutely rigid body (a stamp). The profile of the stamp is described by the function $g(x) = g(-x)$. We require to find the contact stress $p(x)$, and the length 2α of the plane of contact.

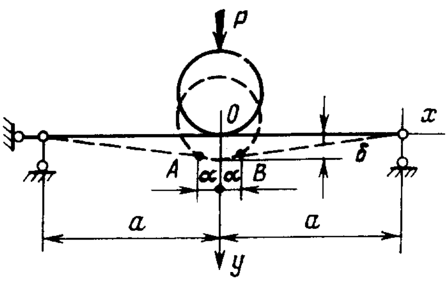

Fig. 1

Using the influence function [3]

$$12aG_a(x, s) = (s-a)(x+a)[(x+a)^2 + (s-a)^2 - 4a^2], \quad x \leqslant s \tag{1.1}$$

(where x must be interchanged with s when $x \geqslant s$) for a hinged beam which represents the Green's function for the boundary value problem

$$Dy^{IV}(x) = q(x) \quad (|x| \leqslant a), \quad y(\pm a) = y''(\pm a) = 0 \tag{1.2}$$

we can write the contact problem formulated above (comp. [2]) in the form of the following integral equation:

$$kp(x) + \frac{1}{D}\int_{-\alpha}^{\alpha} G_a(x, s)\, p(s)\, ds = \delta - g(x) \quad (|x| \leqslant \alpha) \tag{1.3}$$

We find the half-length α of the plane of contact and the translational displacement of the stamp $\delta = y(0)$ from the equations

$$p(\pm\alpha) = 0, \quad \int_{-\alpha}^{\alpha} p(x)\, dx = P \tag{1.4}$$

We note that the Green's function (1.1) can also be written [4] in the following, iterative form:

$$G_a(x, s) = \int_{-a}^{a} G_a^\circ(x, t)\, G_a^\circ(t, s)\, dt \tag{1.5}$$

of the Green's function of the simpler boundary value problem $y''(x) = f(x)$, $|x| \leqslant a$, $y(\pm a) = 0$.

Below we give the examples of solutions of integral equations with kernels in the form of the Green's functions or their iterations.

2. Integral equations on the basic segment. Consider the equation

$$\varphi(x) - \lambda \int_a^b G(x, s)\, \rho(s)\, \varphi(s)\, ds = g(x) \quad (a \leqslant x \leqslant b) \tag{2.1}$$

Here $\rho(x)$ is a specified nonnegative continuous function, $g(x)$ is a specified function and $\varphi(x)$ is an unknown function, the last two functions belonging to $L_2(a, b)$. The kernel $G(x, s)$ is a Green's function of the boundary value problem

$$l[y] = \sum_{j=0}^{n} p_j(x)\, y^{(n-j)}(x) = f(x) \quad (a \leqslant x \leqslant b) \tag{2.2}$$

$$U_\nu[y] = U_{\nu a}[y] + U_{\nu b}[y] = 0 \quad (\nu = 0, 1, \ldots, n-1)$$

$$\left(\{U_{\nu a}[y],\ U_{\nu b}[y]\} = \sum_{j=0}^{n-1} \{A_{\nu j} y^{(j)}(a),\ B_{\nu j} y^{(j)}(b)\}\right)$$

We shall assume, for generality, that the functions $p_j(x)$ and the coefficients $A_{\nu j}$, and $B_{\nu j}$ depend on the parameter λ, consequently $G(x, s)$ also depends on this parameter. The Green's function of the boundary value problem (2.2) exists and is unique [4], provided that the corresponding homogeneous problem has only a trivial solution or, which is equivalent, that the condition

$$\det \{U_\nu[y_j]\}_{\nu, j=0,1,\ldots,n-1} \neq 0 \tag{2.3}$$

(where $(y_j(x)$ is the fundamental system of the operator $l)$ holds.

We have the following representations [5]:

$$G(x, s) = \begin{cases} \sum_{j=0}^{n-1} y_j(x)\, a_j(s) & (a \leqslant x \leqslant s) \\ \sum_{j=0}^{n-1} y_j(x)\, b_j(s) & (s \leqslant x \leqslant b) \end{cases} \tag{2.4}$$

The functions $a_j(s)$ and $b_j(s)$ can be found from the equations

$$\sum_{j=0}^{n-1} (b_j - a_j)\, y_j^{(m)}(s) = 0, \quad \sum_{j=0}^{n-1} (b_j - a_j)\, y_j^{(n-1)}(s) = p_0^{-1}(s) \quad (m = 0, 1, \ldots, n-2) \tag{2.5}$$

$$\sum_{j=0}^{n-1} a_j U_{\nu a}[y_j] + \sum_{j=0}^{n-1} b_j U_{\nu b}[y_j] = 0 \quad (\nu = 0, 1, \ldots, n-1)$$

The integral equation (2. 1) is equivalent [4] to the boundary value problem

$$l[y]-\lambda\rho y=f \quad (a\leqslant x\leqslant b), \quad U_\nu[y]=0 \quad (\nu=0,1,\ldots,n-1) \tag{2.6}$$

under the condition that

$$g(x)=\int_a^b G(x,s)f(s)\,ds \quad (f=l[g]), \quad U_\nu[g]=0 \quad (\nu=0,1,\ldots,n-1) \tag{2.7}$$

The eigenvalues of Eq. (2. 1) coincide [4] with the roots of the equation

$$\det\{U_\nu[y_j(x,\lambda)]\}_{\nu,j=0,1,\ldots,n-1}=0 \tag{2.8}$$

where $y_j(x,\lambda)$ is the fundamental system of the operator $l-\lambda\rho$.

To solve Eq. (2. 1) we must write the boundary value problem (2. 6) with (2. 7) taken into account, in the form

$$l[\varphi-g]-\lambda\rho(\varphi-g)=\lambda\rho g, \quad U_\nu[\varphi-g]=0$$

and use the Green's function $\Gamma_\lambda(x,s)$ of the boundary value problem (2. 6). This yields

$$\varphi(x)=g(x)+\lambda\int_a^b \Gamma_\lambda(x,s)\rho(s)g(s)\,ds \quad (a\leqslant x\leqslant b) \tag{2.9}$$

Using the theory of completely continuous operators [5, 6] we can show (although we shall not attempt it here) that the restrictions (2. 7) imposed on the right-hand side of (2. 1) can be discarded, i. e. that the following theorem holds:

Theorem 2. 1. When the right-hand side of the integral equation (2. 1) is arbitrary and belongs to $L_2(a,b)$, then its unique solution has the form (2. 9) where $\Gamma_\lambda(x,s)$ is the Green's function of the boundary value problem (2. 6). The spectrum of (2. 1) coincides with the roots of (2. 8).

3. Integral equations on a restricted interval. Consider the equation

$$\varphi(x)-\lambda\int_\alpha^\beta G(x,s)\rho(s)\varphi(s)\,ds=g(x) \quad (a<\alpha\leqslant x\leqslant\beta<b) \tag{3.1}$$

First, we shall explain which conditions are satisfied by the Green's function of the boundary value problem (2. 2) at the points $x=\alpha$ and $x=\beta$. From (2. 4) it follows that

$$G^{(m)}(\alpha,s)=\sum_{j=0}^{n-1} y_j^{(m)}(\alpha)a_j(s) \quad (\alpha\leqslant s), \quad m=0,1,\ldots,n-1 \tag{3.2}$$

$$G^{(m)}(\beta,s)=\sum_{j=0}^{n-1} y_j^{(m)}(\beta)b_j(s) \quad (\beta\geqslant s)$$

Since

$$\det\{y_j^{(m)}(x)\}_{j,m=0,1,\ldots,n-1}\neq 0 \quad (\alpha\leqslant x\leqslant\beta)$$

is a Wronskian, from (3. 2) we can find

$$a_j(s)=\sum_{k=0}^{n-1} M_{jk}(\alpha)G^{(k)}(\alpha,s), \quad b_j(s)=\sum_{k=0}^{n-1} M_{jk}(\beta)G^{(k)}(\beta,s)$$

Substituting these expressions into the second equation of (2. 5), we find that

$$U_\nu^{\alpha,\beta}[G(x, s)] = 0 \quad (\nu = 0, 1, \ldots, n-1)$$

where

$$U_\nu^{\alpha,\beta}[y] = \sum_{k=0}^{n-1} S_{\nu k}(\alpha) y^{(k)}(\alpha) + \sum_{k=0}^{n-1} S_{\nu k}(\beta) y^{(k)}(\beta) \tag{3.3}$$

$$\left(S_{\nu k}(\alpha) = \sum_{j=0}^{n-1} M_{jk}(\alpha) U_{\nu a}[y_j], \quad S_{\nu k}(\beta) = \sum_{j=0}^{n-1} M_{jk}(\beta) U_{\nu b}[y_j] \right)$$

If we take into account the fact that the matrices

$$\{y_j^{(m)}(x)\}_{j,m=0,1,\ldots,n-1}, \quad \{M_{jk}(x)\}_{j,k=0,1,\ldots,n-1}$$

are by definition reciprocal, then we can easily show that the equation

$$U_\nu^{\alpha,\beta}[y_j] = U_\nu[y_j]$$

holds. Therefore we have

$$\det\{U_\nu^{\alpha,\beta}[y_j]\}_{\nu,j=0,1,\ldots,n-1} \neq 0$$

and this proves the following theorem:

Theorem 3.1. The Green's function $G(x, s)$ of the boundary value problem (2.2) specified on the interval $[a, b]$ is simultaneously a unique Green's function of the boundary value problem

$$l[y] = f, \quad U_\nu^{\alpha,\beta}[y] = 0 \quad (\nu = 0, 1, \ldots, n-1) \tag{3.4}$$

specified on the interval $[\alpha, \beta]$ contained within $[a, b]$.

From Theorems 2.1 and 3.1 follows

Theorem 3.2. A unique solution of the integral equation (3.1), for arbitrary right-hand side belonging to $L_2(\alpha, \beta)$, is given by the formula

$$\varphi(x) = g(x) + \lambda \int_\alpha^\beta \Gamma_\lambda(x, s)\rho(s) g(s)\, ds \tag{3.5}$$

where the resolvent $\Gamma_\lambda(x, s)$ is a Green's function of the boundary value problem

$$l[y] - \lambda\rho y = f \quad (\alpha \leqslant x \leqslant \beta), \quad U_\nu^{\alpha,\beta}[y] = 0 \quad (\nu = 0, 1, \ldots, n-1) \tag{3.6}$$

The spectrum of the integral equation (3.1) coincides with the roots of the equation

$$\det\{U_\nu^{\alpha,\beta}[y_j(x, \lambda)]\}_{\nu,j=0,1,\ldots,n-1} = 0$$

When the kernel is symmetric, we can use the Hilbert-Schmidt theorem [4, 6, 7] to obtain the solution of the integral equation in the form different from that given by Theorem 3.2. To do this, we denote the orthonormal eigenfunctions of the homogeneous boundary value problem (3.6) by $\varphi_\rho(x, \lambda_n)$. Then

$$\int_\alpha^\beta G(x, s)\rho(s)\varphi_\rho(s, \lambda_n)\, ds = \lambda_n^{-1}\varphi_\rho(x, \lambda_n)$$

$$\int_\alpha^\beta \rho(x)\varphi_\rho(x, \lambda_n)\varphi_\rho(x, \lambda_m)\, dx = \delta_{mn}$$

Taking this into account, we obtain the following formula for the solution of (3.1):

$$\varphi(x) = \sum_{n=0}^{\infty} \frac{\lambda_n f_n}{\lambda_n - \lambda} \varphi_c(x, \lambda_n), \quad f_n = \int_\alpha^\beta f(x) \varphi_c(x, \lambda_n)\, dx$$

It can be shown that Theorems 2.1, 3.1 and 3.2 proved above remain valid also for the systems of integral equations of the type (2.1) or (3.1), but in these cases the part of the scalar boundary value problems and their Green's functions will be played by the matrix [4] boundary value problems and the matrix Green's functions.

4. Integral equations with iterated Green's kernels. Let us consider Eq. (3.1) with the kernel

$$G(x, s) = \int_\alpha^\beta G_0(x, t) G_1(t, s)\, dt \tag{4.1}$$

representing an iteration of two Green's functions each of them corresponding to a different boundary value problem specified on a different interval, each containing the interval $[\alpha, \beta]$. Theorem 3.1 enables us to assume that these Green's functions correspond to the boundary value problems ($j = 0$ and $j = 1$) on the general interval $[\alpha, \beta]$, i.e.

$$l_j[y] = f \quad (\alpha \leqslant x \leqslant \beta), \quad U_{\nu,j}^{\alpha,\beta}[y] = 0 \quad (\nu = 0, 1, \ldots, n_j) \tag{4.2}$$

From (4.1) it follows that $l_0[G] = G_1$ and, that the derivatives of the function $G(x, s)$ of up to the n_0-th order are continuous on the whole of the interval $[\alpha, \beta]$ for any fixed value of s. Moreover, from (4.1) and (4.2) it follows that

$$U_{\nu,0}^{\alpha,\beta}[G] = 0 \ (\nu = 0, 1, \ldots, n_0 - 1), \ U_{\nu,1}^{\alpha,\beta}[l_0 G] = 0 \ (\nu = 0, 1, \ldots, n_1 - 1) \tag{4.3}$$

If we assume that the coefficients of the differential operator l_0 are n_1-times differentiable, then the operator l_1 will be meaningful and we shall be able to consider the boundary value problem

$$l_1 l_0[y] = f \quad (\alpha \leqslant x \leqslant \beta), \quad U_\nu^*[y] = 0 \quad (\nu = 0, 1, \ldots, n_0 + n_1 - 1) \tag{4.4}$$

where

$$U_\nu^*[y] = U_{\nu,0}^{\alpha,\beta}[y] \quad (\nu = 0, 1, \ldots, n_0 - 1), \quad U_{n_0+\nu}^*[y] = U_{\nu,1}^{\alpha,\beta}[l_0 y] \quad (\nu = 0, 1, \ldots, n_1 - 1) \tag{4.5}$$

Its homogeneous variant has only a trivial solution, since

$$\det \{U_\nu^*[y_j]\}_{\nu, j = 0, 1, \ldots, n_0 + n_1 - 1} \neq 0 \tag{4.6}$$

where $y_j(x)$ is the fundamental system of the equation in (4.4), connected with the fundamental systems $y_j^0(x)$ and $y_j^1(x)$ of the differential operators l_0 and l_1 by

$$y_j(x) = y_j^0(x) \quad (j = 0, 1, \ldots, n_0 - 1), \quad y_{n_0+j}(x) = y_j^1(x) \quad (j = 0, 1, \ldots, n - 1) \tag{4.7}$$

and we also have

$$l_0[y_j^1] = y_j^0 \quad (j = 0, 1, \ldots, n_1 - 1)$$

To confirm the validity of (4.6), we take into account the fact that the matrix appearing in the expression is, by virtue of (4.7) and (4.5), a right triangular block matrix consisting of four blocks, and the diagonal blocks will be represented by the matrices

$$\{U_{\nu,0}^{\alpha,\beta}[y_j^0]\}_{\nu, j = 0, 1, \ldots, n_0 - 1}, \quad \{U_{\nu,1}^{\alpha,\beta}[y_j^1]\}_{\nu, j = 0, 1, \ldots, n_1 - 1}$$

with, by virtue of Theorem 3.1, nonzero determinants. The determinant (4.6) is equal to the product of these determinants [7].

Consequently, if a Green's function of the problem (4.4) exists, then it is unique [4]. But, according to (4.1) and (4.3) the function

$$y(x) = \int_\alpha^\beta G(x, s) f(s)\, ds$$

is a solution of the boundary value problem (4.4), therefore the formula (4.1) defines its Green's function. This proves, with Theorem 2.1 taken into account, the following theorem.

Theorem 4.1. *A unique solution of the integral equation (3.1) belonging to $L_2(\alpha, \beta)$ with a kernel of the type (4.1), is determined (with $g \in L_2$) by the formula (3.4) in which $\Gamma_\lambda(x, s)$ is the Green's function of the boundary value problem*

$$l_1 l_0 [y] - \lambda\rho y = f \quad (\alpha \leqslant x \leqslant \beta), \quad U_\nu^* [y] = 0 \quad (\nu = 0, 1, \ldots, n_0 + n_1 - 1) \tag{4.8}$$

The spectrum is determined here by the roots of the equation

$$\det \{U_\nu^* [y_j(x, \lambda)]\}_{\nu, j = 0, 1, \ldots, n_0 + n_1 - 1}$$

where $y_j(x, \lambda)$ is the fundamental system of the solution of the operator $l_1 l_0 - \lambda\rho$.

In a particular case when

$$G(x, s) = \int_\alpha^\beta G_0(x, t) G_0(t, s)\, dt \tag{4.9}$$

we can show another form of solution (under the assumption that G_0 is independent of λ). To do this, we set in (3.1) with the kernel (4.9) $\lambda = \mu^2$ and denote the resolvent of the equation

$$\varphi(x) - \mu \int_\alpha^\beta G_0(x, s) \varphi(s)\, ds = g(x) \quad (\alpha \leqslant x \leqslant \beta)$$

by $\Gamma_0(x, s, \mu)$. Then the resolvent $\Gamma(x, s, \mu^2)$ of the integral equation considered can be written in the form [3]

$$2\mu\Gamma(x, s, \mu^2) = \Gamma_0(x, s, \mu) - \Gamma_0(x, s, -\mu)$$

and we have

Theorem 4.2. *A unique solution belonging to $L_2(\alpha, \beta)$ of the integral equation (3.1), for $\lambda = \mu^2$, with the kernel (4.9) is given (with $\rho \equiv 1$) by the formula*

$$\varphi(x) = g(x) + \frac{\mu}{2} \int_\alpha^\beta [\Gamma_0(x, s, \mu) - \Gamma_0(x, s - \mu)]\, g(s)\, ds$$

where $\Gamma_0(s, x, \mu)$ is the Green's function of the boundary value problem

$$l_0 [y] - \mu y = f \quad (\alpha \leqslant x \leqslant \beta), \quad U_{\nu,0}^{\alpha,\beta} [y] = 0 \quad (\nu = 0, 1, \ldots, n_0 - 1)$$

The eigenvalues are equal to squares of the roots of the equation

$$\det \{U_{\nu,0}^{\alpha,\beta} [y_j^0(x, \mu)]\}_{\nu, j = 0, 1, \ldots, n_0 - 1} = 0$$

where $y_j^0(x, \mu)$ is the fundamental system of the operator $l_0 - \mu$.

5. Integral equations with superposition of Green's functions.

Equations of the form

$$\varphi - \lambda(G_0\varphi + G_1\varphi) = g \quad (\alpha \leqslant x \leqslant \beta) \tag{5.1}$$

$$G_j\varphi = \int_\alpha^\beta G_j(x, s)\rho_j(s)\varphi(s)\,ds \quad (j = 0, 1)$$

arise when two thin-walled elements are in contact with each other. To construct their solution we introduce the functions

$$\chi_0 = \varphi - \lambda G_0\varphi, \quad \chi_1 = \varphi - \lambda G_1\varphi \tag{5.2}$$

and rewrite (5.1) in the form

$$\chi_0 + \chi_1 - \varphi = g \tag{5.3}$$

if

$$\det\{U_{\nu,m}^{\alpha,\beta}[y_j^m(x, \lambda)]\}_{\nu,j=0,1,\ldots,n_m-1} \neq 0 \qquad (m = 0,1) \tag{5.4}$$

then by Theorem 2.1 we can transform Eqs. (5.2) into

$$\varphi = \chi_0 + \lambda\Gamma_0\chi_0, \quad \varphi = \chi_1 + \lambda\Gamma_1\chi_1 \tag{5.5}$$

$$\left(\Gamma_m f = \int_\alpha^\beta \Gamma_\lambda^m(x,s) f(s)\,ds,\ m = 0,1\right)$$

where $\Gamma_\lambda^m(x, s)$ are the Green's functions of the boundary value problems

$$l_m[y] - \lambda\rho_m y = f \quad (\alpha \leqslant x \leqslant \beta), \quad U_{\nu,m}^{\alpha,\beta}[y] = 0 \tag{5.6}$$

$$(\nu = 0,1,\ldots, n_m - 1),\ m = 0,1$$

where $(y_j^m(x, \lambda)$ are the fundamental systems of the operators $l_m - \lambda\rho_m)$. Substituting the consecutive values of φ given by (5.5) into (5.3), we arrive at the following system of integral equations:

$$\chi_0 - \lambda\Gamma_1\chi_1 = g, \quad \chi_1 - \lambda\Gamma_0\chi_0 = g \tag{5.7}$$

We shall show that if a solution of this system has been constructed, then the solution of the initial equation (5.1) will be given by one of the formulas of (5.5). Let us e.g. substitute the first expression for φ from (5.5) into the initial equation (5.1) and take into account the fact that the known integral equations satisfied by the resolvent kernels [3, 5, 6] will, in this case, have the form

$$\lambda\Gamma_0 - \lambda G_0 = \lambda^2 G_0\Gamma_0, \quad \lambda\Gamma_1 - \lambda G_1 = \lambda^2 G_1\Gamma_1 \tag{5.8}$$

As the result we obtain, in place of (5.1), $\chi_0 - \lambda G_1\chi_0 - \lambda^2 G_1\Gamma_0\chi_0 = g$. To confirm that the above expression is an identity, we substitute into it the expressions

$$\lambda G_1\chi_0 = \lambda^2 G_1\Gamma_1\chi_1 + \lambda G_1 g_1, \quad \lambda^2 G_1\Gamma_0\chi_0 = \lambda G_1\chi_1 - \lambda G_1 g$$

obtained from Eqs. (5.7) by operating on them with G_1, and take into account the first equation of (5.7).

On the other hand, any solution of (5.1) will lead, through the formulas (5.2), to a solution of the system (5.7). To show this it is sufficient to substitute (5.2) into (5.7) and use (5.8).

Let us now turn our attention to the problem of solving the system (5.7). Clearly, any solution of this system will also be a solution of the following two independent equations:

$$\chi_0 - \lambda^2\Gamma_1\Gamma_0\chi_0 = g + \lambda\Gamma_1 g, \quad \chi_1 - \lambda^2\Gamma_0\Gamma_1\chi_1 = g + \lambda\Gamma_0 g \tag{5.9}$$

obtained by eliminating from the system χ_1 (first equation) and χ_0 (second equation). We shall show that the converse is also true, i.e. that the solutions χ_0 and χ_1 of the integral equation (5.9) represent the solution of the system (5.7), provided that λ is not an eigenvalue of the integral equations (they have a continuous spectrum [3, 5]). In fact operating on the first equation of (5.9) with Γ_0, we convert it into the form

$$g + \lambda\Gamma_0\chi_0 - \lambda^2\Gamma_0\Gamma_1 (g + \lambda\Gamma_0\chi_0) = g + \lambda\Gamma_0 g$$

Comparing this equation with the second equation of (5.9) we conclude, that $\chi_1 - \lambda\Gamma_0\chi_0 = g$. We obtain the second equation of (5.7) in a similar manner, by operating on the second equation of (5.9) with Γ_1.

Thus we arrive at the solution of one of the equations of (5.9), e.g. of the second equation. To solve it, we utilize the results of Sect. 4. The part of the kernels $G_0(x, s)$ and $G_1(x, s)$ is played by the kernels $\Gamma_\lambda{}^0(x, s)$ and $\Gamma_\lambda{}^1(x, s)$, and the boundary value problem (4.8) assumes the form

$$\begin{aligned}&(l_1 - \lambda\rho_1)(l_0 - \lambda\rho_0)[y] - \lambda^2 y = f \quad (\alpha \leqslant x \leqslant \beta) \\ &U_\nu'[y] = 0 \quad (\nu = 0, 1, \ldots, n_0 + n_1 - 1)\end{aligned} \tag{5.10}$$

where
$$U_\nu'[y] = U_{\nu,0}^{\alpha,\beta}[y] \quad (\nu = 0, 1, \ldots, n_0 - 1)$$

$$U'_{\nu+n_0}[y] = U_{\nu,0}^{\alpha,\beta}[l_0 y] - \lambda U_{\nu,1}^{\alpha,\beta}[\rho_0 y] \quad (\nu = 0, 1, \ldots, n_1 - 1)$$

It is clear that the differential operator introduced here has a meaning only when the function $\rho_0(x)$ is at least n_1-times continuously differentiable.

As previously, we denote the fundamental system of the differential equation of (5.10) by $y_j(x, \lambda)$, $j = 0, 1, \ldots, n_0 + n_1 - 1$. Then, in accordance with Sect. 4, the condition

$$\det \{U_\nu'[y_j(x, \lambda)]\}_{\nu, j = 0, 1, \ldots, n_0 + n_1 - 1} \neq 0 \tag{5.11}$$

will ensure that a unique Green's function $\Gamma(x, s, \lambda)$ of the boundary value problem (5.10) can be constructed, and hence the integral equation of (5.9). This proves the following theorem:

Theorem 5.1. If $\rho_0(x)$ in the integral equation (5.1) is n_1-times continuously differentiable and the parameter λ is such that the conditions (5.11) and (5.4) hold, then the solution of this equation can be obtained by the formulas

$$\varphi(x) = \chi_1(x) + \lambda \int_\alpha^\beta \Gamma_\lambda{}^1(x, s)\chi_1(s)\, ds \tag{5.12}$$

$$\chi_1(x) = g(x) + \int_\alpha^\beta \left[\Gamma(x, s, \lambda) + \Gamma_\lambda{}^0(x, s) + \lambda^2 \int_\alpha^\beta \Gamma(x, t, \lambda)\Gamma_\lambda{}^0(t, s)\, dt\right] g(s)\, ds$$

6. Integral equations on several intervals. Let us consider the equation on two intervals

$$\varphi(x) - \lambda \left(\int_\alpha^\omega + \int_\rho^\beta\right) G(x, s)\rho(s)\varphi(s)\, ds = g(x) \tag{6.1}$$

$$x \in [\alpha, \omega] + [\rho, \beta], \quad a \leqslant \alpha < \omega < \rho < \beta \leqslant b$$

By Theorem 3.1 the kernel of this equation is also the Green's function of the boundary value problem (3.4). Assuming that

$$\varphi(x) \equiv 0, \quad \omega < x < \rho \tag{6.2}$$

we define additionally Eq. (6.1) on the whole interval $[\alpha, \beta]$ in the following manner:

$$\varphi(x) - \lambda \int_\alpha^\beta G(x, s)\rho(s)\varphi(s)\,ds = h(x) + \psi(x), \quad x \in [\alpha, \beta] \tag{6.3}$$

$$h(x) = g(x), \quad \psi(x) \equiv 0, \quad x \in [\alpha, \omega] + [\rho, \beta]$$

$$h(x) = 0, \psi(x) = -\lambda \int_\alpha^\beta G(x, s)\rho(s)\varphi(s)\,ds, \quad x \in [\omega, \rho]$$

The function $\rho(x)$ is continued into the interval $[\omega, \rho]$ in an arbitrary manner, provided that its continuity is preserved.

If we assume for the time being that the function $\psi(x)$ is known and λ is different from the eigenvalue, then by Theorem 3.2 the integral equation (6.3) will have a unique solution in the form of the function

$$\varphi(x) = h(x) + \psi(x) + \lambda \int_\alpha^\beta \Gamma_\lambda(x, s)[h(s) + \psi(s)]\rho(s)\,ds, \; x \in [\alpha, \beta] \tag{6.4}$$

The resolvent $\Gamma_\lambda(x, s)$ will be a Green's function of the boundary value problem (3.6) and the spectrum of the integral equation (6.3) will be determined by the zeros of the determinant

$$\det\{U_\nu^{\alpha,\beta}[y_j(x, \lambda)]\}_{\nu, j=0,1,\ldots,n-1} \tag{6.5}$$

Taking now into account in (6.4) the relations (6.2), we arrive at the following equation for $\psi(x)$:

$$\psi(x) + \lambda \int_\omega^\rho \Gamma_\lambda(x, s)\rho(s)\psi(s)\,ds = r(x) \quad (\omega \leqslant x \leqslant \rho) \tag{6.6}$$

$$r(x) = -\lambda\left(\int_\alpha^\omega + \int_\rho^\beta\right)\Gamma_\lambda(x, s)\rho(s)g(s)\,ds$$

According to Theorem 3.1, when the values of λ differ from those of the roots of the determinant (6.5), then its kernel $\Gamma_\lambda(x, s)$ will be a Green's function of the boundary value problem

$$l[y] - \lambda\rho y = f \; (\omega \leqslant x \leqslant \rho), \; U_\nu^{\omega,\rho}[y] = 0 \; (\nu = 0, 1, \ldots, n-1) \tag{6.7}$$

and we also have

$$U_\nu^{\omega,\rho}[y] = \sum_{k=0}^{n-1} S_{\nu k}(\omega, \lambda)y^{(k)}(\omega) + \sum_{k=0}^{n-} S_{\nu k}(\rho, \lambda)y^{(k)}(\rho) \tag{6.8}$$

$$\begin{bmatrix} S_{\nu k}(\omega, \lambda) \\ S_{\nu k}(\rho, \lambda) \end{bmatrix} = \sum_{j=0}^{n-1}\begin{bmatrix} M_{jk}(\omega, \lambda) \\ M_{jk}(\rho, \lambda) \end{bmatrix}\sum_{l=0}^{n-1}\begin{bmatrix} s_{\nu l}(\alpha) & y_j^{(l)}(\alpha, \lambda) \\ s_{\nu l}(\beta) & y_j^{(l)}(\beta, \lambda) \end{bmatrix}$$

The matrix $\{M_{jk}(x, \lambda)\}_{j,k=0,1,\ldots,n-1}$ is an inverse of the matrix

$$\{y_k^{(j)}(x, \lambda)\}_{j,k=0,1,\ldots,n-1},$$

the determinant of which is a Wronskian of the differential equation appearing in (6.7), and is therefore different from zero. The coefficients $s_{\nu l}(\alpha)$ and $s_{\nu l}(\beta)$ are determined, as before, by the formulas (3.3).

Thus the solution of the integral equation (6. 6) requires, according to Theorem 2. 1, constructing a Green's function of the boundary value problem

$$l[y] = f \quad (\omega \leqslant x \leqslant \rho), \quad U_\nu^{\omega,\rho}[y] = 0 \quad (\nu = 0, 1, \ldots, n-1) \tag{6.9}$$

If λ is such that the condition

$$\det\{U_\nu^{\omega,\rho}[y_j]\}_{\nu,j=0,1,\ldots,n-1} \neq 0 \tag{6.10}$$

holds, we can use formulas of the type (2. 4) and (2. 5) to construct a unique Green's function $R_\lambda(x, s)$ of the boundary value problem (6. 9) and obtain the solution of the integral equation (6. 6) in the form

$$\psi(x) = r(x) - \lambda \int_\omega^\rho R_\lambda(x, s)\rho(s) r(s)\, ds \quad (\omega \leqslant x \leqslant \rho) \tag{6.11}$$

To obtain the initial equation (6. 1) it remains to substitute (6. 11) into (6. 4), and take into account (6. 3) and the equations of the type (5. 8) for the resolvent R_λ. As the result, we have

$$\varphi(x) = g(x) - r(x) + \lambda \int_\omega^\rho R_\lambda(x, s)\rho(s) r(s)\, ds \tag{6.12}$$

$$x \in [\alpha, \omega] + [\rho, \beta], \; r(x) = -\lambda\left(\int_\alpha^\omega + \int_\rho^\beta\right)\Gamma_\lambda(x, s)\, g(s) ds$$

and the result obtained can be stated in the form of the following assertion:

Theorem 6. 1. If the conditions (6. 5) and (6. 10) hold, then the solution of Eq. (6. 1) is given by the formula (6. 12). The functions $\Gamma_\lambda(x, s)$ and $R_\lambda(x, s)$ are Green's functions of the boundary value problems (3. 6) and (6. 9), respectively.

It can also be deduced from the proof that the spectrum of the integral equation is contained within the set of zeros of the determinants (6. 5) and (6. 10).

The above method of solving Eq. (6. 1) can obviously be generalized to the case when the number of intervals of integration is greater than two. Such equations correspond to the problems with several regions of contact (e. g. impressing several stamps into a beam, etc.).

7. An example of application of the results obtained. Using Theorem 3. 2 we can write the solution of the integral equation (1. 3) to which the contact problem discussed in Sect. 1 (Fig. 1) was reduced. We can, however, obtain simpler formulas by changing slightly the mathematical formulation of the problem. With this purpose in mind, we shall analyze the displacements of the points on the beam surface in the zone of contact, beginning from the instant of termination of the contact. We place fictitious supports at the end points A and B (Fig. 2) of the contact interval $-\alpha \leqslant x \leqslant \alpha$. The displacements of the beam on the segment indicated will consist of the deflections $y_p(x)$ of a hinged beam (of length 2α) caused by the contact pressure $p(x)$, and of the deflections $y_M(x)$

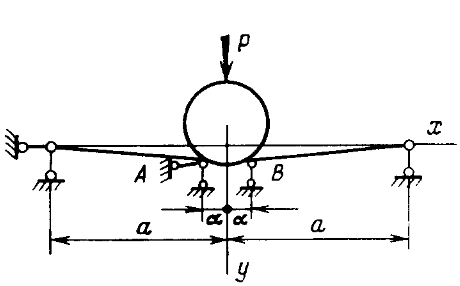

Fig. 2

due to the end moments $M = {}^1/_2 P\beta$ ($\beta = a - \alpha$), equal to the bending moments at the cross sections $x = \pm\alpha$ of the initial beam (of length $2a$). The displacements of the points of the stamp which have come into contact, will be expressed by the formula

$$h(x) = \delta_0 - g(x), \quad \delta_0 = \delta - y(\alpha)$$

and $h(\alpha) = 0$, i.e. $\delta_0 = g(\alpha)$. Consequently, the condition of contact is

$$kp(x) + y_p(x) + y_M(x) = g(\alpha) - g(x) \tag{7.1}$$

Taking into account the equations

$$4Dy_M(x) = P\beta(\alpha^2 - x^2), \quad Dy_p(x) = \int_{-\alpha}^{\alpha} G_\alpha(x, s)\, p(s)\, ds$$

we obtain, in place of (1.3), the integral equation

$$p(x) + c^2 \int_{-\alpha}^{\alpha} G_\alpha(x, s)\, p(s)\, ds = f(x), \quad |x| \leqslant \alpha \tag{7.2}$$

$$c^2 = (kD)^{-1}, \quad f(x) = k^{-1}[g(\alpha) - g(x)] - {}^1/_4 P\beta c^2(\alpha^2 - x^2)$$

the kernel of which is determined by the formula (1.1) in which a is replaced by α. Equation (7.2) represents a particular case of an equation the solution of which is given by Theorem 5.1 and has the form

$$p(x) = f(x) + c\,\mathrm{Im}\left[\int_{-\alpha}^{\alpha} \Gamma_0(x, s, ic)\, f(s)\, ds\right] \tag{7.3}$$

$$\nu\Gamma_0(x, s, ic) = \sin\nu(\alpha + x)\sin\nu(\alpha - s)\,\mathrm{cosec}\, 2\nu\alpha$$
$$(x \leqslant s,\ \nu = \sqrt{{}^1/_2 c}\,(1 + i))$$

(where x and s must be interchanged when $x \geqslant s$). Here $\Gamma_0(x, s, ic)$ is the Green's function of the boundary value problem

$$-y''(x) - \nu^2 y(x) = f(x) \quad (|x| \leqslant \alpha, \quad \nu^2 = ic), \quad y(\mp\alpha) = 0$$

and the solution obtained satisfies automatically the first condition of (1.4).

If we assume, as we usually do in the contact problems [8], that $g(x) = \gamma x^2$, then the formula (7.3) can be reduced to the following simple expression:

$$\frac{cp(x)}{2C_p} = \frac{\omega^2 - \xi^2}{2} + \frac{\sin{}^1/_2(\omega - \xi)\,\mathrm{sh}{}^1/_2(\omega + \xi) + \mathrm{sh}{}^1/_2(\omega - \xi)\sin{}^1/_2(\omega + \xi)}{\cos\omega + \mathrm{ch}\,\omega} \tag{7.4}$$

$$(C_p = k^{-1}\gamma - {}^1/_4 P\beta c^2,\ \omega = \alpha\sqrt{2c},\ \xi = x\sqrt{2c})$$

The length of the region of contact must be obtained from the equation

$$\frac{P}{C_p} = \frac{8}{3}\alpha^3 + \frac{4}{\sqrt{2}c^{3/2}}\frac{\mathrm{sh}\,\omega - \sin\omega}{\cos\omega + \mathrm{ch}\,\omega}$$

which represents the outcome of the second condition of (1.4) with (7.4) taken into account.

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MATRIX FACTORIZATION METHOD IN MIXED STATIC PROBLEMS OF AN ELASTIC WEDGE

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We study the plane contact static problems of an elastic wedge under the condition that it is rigidly connected with the stamp. The question of solvability of the above problems is investigated and an approximate method of solution involving the matrix factorization method developed in paper [1] and others, is proposed. The plane problems of elasticity for a wedge with discontinuous boundary conditions were investigated by a number of authors. For example, in [2] the author used the method of reduction to an integral Wiener-Hopf equation to investigate the problem of indenting a rigid stamp into a perfectly smooth face of an elastic wedge. In [3] a similar problem was reduced to a certain Fredholm equation of second kind. In [4] and others (*), the asymptotic and orthogonal polynomial methods were successfully used.

1. We shall consider an elastic wedge, the upper face of which is acted upon by a strip stamp rigidly adhering to the face. The boundaries of the zone of contact are separated from the edge of the wedge by the distances a^* and b^*, respectively ($b^* > a^* > 0$), and we study the case of plane deformation. For a displacement vector $\mathbf{u}^\circ(r) = \{u_r^\circ(r), u_\varphi^\circ(r)\}$ defined in the region $\varphi = \alpha$, $a^* < r < b^*$, we require to find the total stress vector $\sigma^\circ(r) = \{\tau_{\varphi r}^\circ(r), \sigma_\varphi^\circ(r)\}$ in the zone of contact for each of the following conditions of clamping of the lower edge of the wedge ($\varphi = 0$, $0 < r < \infty$):

$$\mathbf{A} \quad u_r(r, 0) = u_\varphi(r, 0) = 0$$

$$\mathbf{B} \quad \tau_{\varphi r}(r, 0) = \sigma_\varphi(r, 0) = 0$$

*) Lutchenko, S. A. and Popov, G. Ia. On certain plane contact problems of the theory of elasticity for a wedge. In coll.: 3-rd All-Union Convention on Theoretical and Applied Mechanics. Moscow, 1968. Annot. dokl., Moscow, "Nauka", 1968.